Some Paradoxes of Computation in Mathematical Economics

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> "This 'next step in [economic] analysis,' conjectured the *doyen* of mathematical economics, Kenneth Arrow ([3], p.S398), '[would be] a more consistent assumption of computability in the formulation of economic hypotheses.' But this has *not* been taken by economic theorists..." [4]

0. PHILOSOPHIES OF MATHEMATICS [5: 4]

1. Mathematics is about some independently existing abstract "Platonic" reality, the truths of which are objective and graspable through the employment of pure reason
2. Mathematics is the objective part of human conceptions and constructions; whether or not there are essential limits to the power of human reasoning cannot be answered at this time

I. SOLVABLE AND UNSOLVABLE PROBLEMS

In 1900, after the birth of mathematical economics, Hilbert gave a dramatic address in Paris at the 2nd International Congress of Mathematicians, stating that every and all problems are solvable, or what he called *the axiom of solvability*. "There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no *ignoramibus*." [5]

In the 1930s, mathematical discoveries by Kurt Gödel, Emil Post, Alan Turing, and Alonzo Church turned Hilbert's axiom of solvability on its head. There were indeed unsolvable problems: in classical real analysis, paradoxes such as the unprovability of "this statement is false;" in computable analysis, an unsolvable problem was a problem for which a Turing machine computing the solution would never halt.

II. DOES ECONOMICS NEED TO BE ALGORITHMIC?

Some mathematicians claim that mathematics doesn't make sense as a tool for humans to study their reality if we cannot calculate in it (Wittgenstein). Real analysis underpinned by Zermelo-Fraenkel set theory plus the axiom of choice is empirically meaningless, in that you do not need to develop a method for solving a problem in order to claim it "can be solved." Real systems calculate, and any applied work is calculation. Thus, real solutions need to be calculable.

Many of the questions posed in traditional mathematical economics are formally meaningless in computable analysis. They result in **recursive undecidabilities**, **algorithmic unsolvabilities**, and **uncomputabilities**, and **formal incompleteness**.

Yet, the framework and results of traditional mathematical economics form "the benchmark and the ideal, respectively, to which agents and economic systems must conform on pain of giving rise to irrationalities and inefficiencies" (4: 6).

III. LIST OF RESULTS

First six are from [4]:

1. Nash equilibria of finite games are constructively indeterminate (Velupillai; Papadimitriou)
2. Computable General Equilibria are neither computable nor constructive (Velupillai)
3. The Two Fundamental Theorems of Welfare Economics are Nonconstructive and Uncomputable, respectively (Velupillai)
4. There is no effective procedure to generate preference orderings (Velupillai)
5. Recursive Competitive equilibria, underpinning the RBC model and, hence, the Stochastic Dynamic General Equilibrium benchmark model of Macroeconomics, are uncomputable.
6. There are games in which the player who in theory can always win cannot do so in practice because it is impossible to supply him / her with effective instructions regarding how he / she should play in order to win. (Rabin: see proof [9: 71])

Many of the results are derived from recasting traditional economic questions in alternative logical and mathematical frameworks.

IV. ALTERNATIVE MATHEMATICAL FRAMEWORKS

1. classical real analysis
2. constructive analysis
3. computable analysis
4. non-standard analysis

1. *Classical real analysis*: Axiomatic mathematics of Russell and Whitehead's *Mathematica Principia*.
    a. Zermelo-Fraenkel axioms, including the Axiom of Choice, hold in this framework.

2. *Constructive mathematics*: "Constructive mathematics is distinguished from its traditional counterpart, classical mathematics, by the strict interpretation of the phrase "there exists" as "we can construct". In order to work constructively, we need to re-interpret not only the existential quantifier but all the logical connectives and quantifiers as instructions on how to construct a proof of the statement involving these logical expressions." [1]

3. *Computable analysis*: “A mathematical problem is *computable* if it can be solved in principle by a computing device. Some common synonyms for “computable” are “solvable”, “decidable”, and “recursive.”” [2]
    a. Questions of computability usually relate to whether a solution is computable in principle and/or in practice.
    b. Whether a solution is computable *in principle* can be answered by whether or not an ideal computer (Turing Machine) could solve it, in principle.
    c. Whether a solution is computable *in practice* is answerable by how long the computation will take. For instance, in the 1930s Tarski established a decision procedure for the algebra of real numbers; however, in the 1970s Fischer and Rabin showed that no algorithm for the algebra of real numbers can work faster than the exponential rate in general.

V. ALTERNATIVE MATHEMATICAL LOGICAL SYSTEMS

1. set theory
2. proof theory
3. recursion theory
4. model theory

VI. MATHEMATICAL ECONOMICS: Currently, apart from the tiny group doing computable economics, takes the form of **classical real analysis & set theory**. Why? Possibly a combination or historical and policy-exigent reasons. There does not appear to be a good *scientific* reason why mathematical economics is done entirely in classical real analysis & set theory.

1. Mirowski (2002) suggests that mathematical economics may get its deterministic nature from its development just before the development of the Second Law of Thermodynamics.
    a. “We may safely accept as a satisfactory scientific hypothesis that the doctrine so grandly put forward by Laplace, who asserted that a perfect knowledge of the universe, as it existed at any moment, would give perfect knowledge of what was to happen thenceforth and forever after. Scientific inference is impossible, unless we regard the present as the outcome of what is past, and the cause of what is to come. To the perfect intelligence nothing is uncertain” [(Jevons) 8: 738-39]
2. I suggest that the hopes pinned on mathematical economics as an instrument with which to control and plan the economy were raised before the economists absorbed the Gödel-Post-Church-Turing undecidability results.

VII. MATHEMATIZING ECONOMICS

Economics should be mathematized to be useful in formalizing economic concepts and entities with a view to application; that is, the tools of mathematical economics need to have numerical and computational content.

BACK MATTER

SOME AXIOMS AND UNCOMPUTABLE AND NON-CONSTRUCTIVE MATH IN DEBREU'S *THEORY OF VALUE*

1. The Heine-Borel Theorem (uncomputable)
2. The Bolzano-Weierstrauss Theorem (non-constructive)
3. The Completeness Theorem (axiom of choice)
4. Specker's Theorem (computable with heavy caveats)
5. The Hahn-Banach Theorem (uncomputable and non-constructive in its classical form)
6. Brouwer's Fixed Point Theorem and Kakutani's Fixed Point Theorem (non-constructive)
7. "There are clear intuitive notions of continuity which cannot be [topologically] defined" [4: 12 citing 7: 342]

DEFINITIONS

algorithm: A step-by-step procedure or sequence of rules to go from the data of any specific problem of a certain type to its solution. The data could be a number, an expression, a sequence of numbers and symbols, etc.

completeness: An axiom system *T* is complete if every proposition in the subject matter can be proved in *T*.

consistency: A system or statement is consistent if a model of it can be produced, by Gödel's Incompleteness Theorem. That is, a theory *T* (a set of axioms) is inconsistent if there is a proof in *T* of a formula *and* its negation.

independence: statement *A* is independent from a set of statements *S* if the negation of *A* is consistent with *S*.

meta-mathematics: The consideration of the consistency, completeness, and independence of axioms within a mathematical framework

The Axiom of Choice: One of the Zermelo-Fraenkel axioms, which implies that for any set there exists a well-ordering of that set. However, the axiom doesn't tell you *how to construct* the well-ordering of that set.

Turing Machine: An ideal computer with no restrictions placed on how much time and memory space are required to carry out any given computation. A Turing Machine can compute what can be computed in principle.

undecidable/uncomputable/unsolvable problem: No possible algorithm can be used to determine the validity of the solution to the problem in a given mathematical framework.

REFERENCES

[1] Bridges, Douglas and Palmgren, Erik, "Constructive Mathematics", *The Stanford Encyclopedia of Philosophy* (Winter 2013 Edition), Edward N. Zalta (ed.), http://plato.stanford.edu/archives/win2013/entries/mathematics-constructive/

[2] Immerman, Neil, "Computability and Complexity", *The Stanford Encyclopedia of Philosophy* (Spring 2016 Edition), Edward N. Zalta (ed.), http://plato.stanford.edu/archives/spr2016/entries/computability/

[3] Arrow, K. J. (1986). Rationality of self and others in an economic system. *Journal of Business*, S385-S399.

[4] Velupillai, K. Vela (2012). *Computable Foundations for Economics* (Routledge Advances in Experimental and Computable Economics). Taylor and Francis.

[5] Feferman, S. (1994). Deciding the undecidable: Wrestling with Hilbert's problems. *In the light of logic*.

[6] *For a thorough mathematical introduction to these topics, complete chapters 1 – 3, 8 in* Kleene's Introduction to Meta-Mathematics, *1980.*

[7] Gandy, Robin (1995), *The Confluence of Ideas in 1936, The Universal Turing Machine – A Half-century Survey (Second Edition),* edited by Rolf Herken, Springer-Verlag, Wien and New York, 1995, pp. 51– 102.

[8] Jevons, William Stanley (1905). *The Principles of Science*. 2nd ed. London: Macmillan.

[9] Lewis, A. A. (1985). On effectively computable realizations of choice functions: Dedicated to Professors Kenneth J. Arrow and Anil Nerode. *Mathematical Social Sciences*, *10*(1), 43-80.